Question 4: Solve the following system of linear differential equations.

$$\begin{aligned} \frac{d}{dt}x(t) &= 2x(t) + 5y(t), \\ \frac{d}{dt}y(t) &= 3x(t) + 4y(t), \end{aligned}$$

with initial values $x(0) = 17, y(0) = -7$.

Solution: The eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

are $\lambda = -1$ and $\mu = 7$ with corresponding eigenvectors

$$v = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The initial conditions give the linear system

$$\begin{bmatrix} -5 \\ 3 \end{bmatrix} C_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} C_2 = \begin{bmatrix} 17 \\ -7 \end{bmatrix}.$$

This system has the solution $C_1 = -3, C_2 = 2$. Hence the solution to the differential equation is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -3e^{-t} \begin{bmatrix} -5 \\ 3 \end{bmatrix} + 2e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

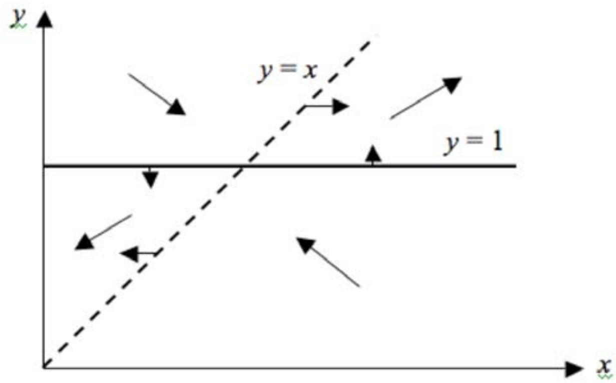
Question 5: Consider the following system of differential equations:

$$\begin{aligned} \frac{d}{dt}x(t) &= y(t) - 1, \\ \frac{d}{dt}y(t) &= x(t) - y(t). \end{aligned}$$

(a) Graph the nullclines in the phase plane.
(b) Draw the direction arrows in each region in the phase plane and on each part of the nullclines.
(c) Find the equilibrium point(s).

Solution: The x-nullcline is the set of points there $dx/dt = 0$, i.e., $y - 1 = 0$ or $y = 1$. This is a horizontal line. The y-nullcline is the set of points where $dy/dt = 0$, i.e., $x - y = 0$ or $y = x$, which is the 45 degree line. The equilibrium points are the intersections of the nullclines. In this case there is only one point, which is $(x, y) = (1, 1)$.

For the direction arrows on the x-nullcline: Since we are on the x-nullcline, the arrows are either pointing up or down. We calculate $dy/dt > 0$ if and only if $x > y$. Similarly in all the other places. The phase plane is given in the following plot:



Question 6: Consider the following competition system:

$$\begin{aligned}\frac{d}{dt}x &= x(10 - 4x - 2y),\\ \frac{d}{dt}y &= 2y(3 - y - x).\end{aligned}$$

(a) Graph the nullclines in the phase plane.
(b) Draw the direction arrows in each region in the phase plane and on each part of the nullclines.
(c) Find the four equilibrium points.
(d) Find the Jacobi matrix at the coexistence equilibrium and determine its eigenvalues. Is the equilibrium stable?

Solution: The x-nullcline is given by the condition $dx/dt = 0$. Hence, either

$$x = 0, \quad \text{or} \quad y = 5 - 2x.$$

The y-nullcline is given by $dy/dt = 0$, or explicitly by

$$y = 0, \quad \text{or} \quad y = 3 - x.$$

There are four equilibrium points, namely the intersections of the nullclines, and these are

$$E_1 = (0,0), \quad E_2 = (5/2, 0), \quad E_3 = (0,3), \quad E_4 = (2,1).$$

To find the Jacobi matrix, we have to differentiate the first and the second function on the right hand side in the equation with respect to x and y respectively. We get

$$J(x,y) = \begin{bmatrix} 10 - 8x - 2y & -2x \\ -2y & 6 - 4y - 2x \end{bmatrix}.$$

At the coexistence point $E_4 = (2, 1)$, this Jacobi matrix is

$$J(2,1) = \begin{bmatrix} -8 & -4 \\ -2 & -2 \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda = -5 \pm \sqrt{17}.$$

Since $17 < 25$ we have $\sqrt{17} < 5$, and hence the eigenvalues are both negative. Therefore, E_4 is stable, see Figure 1.

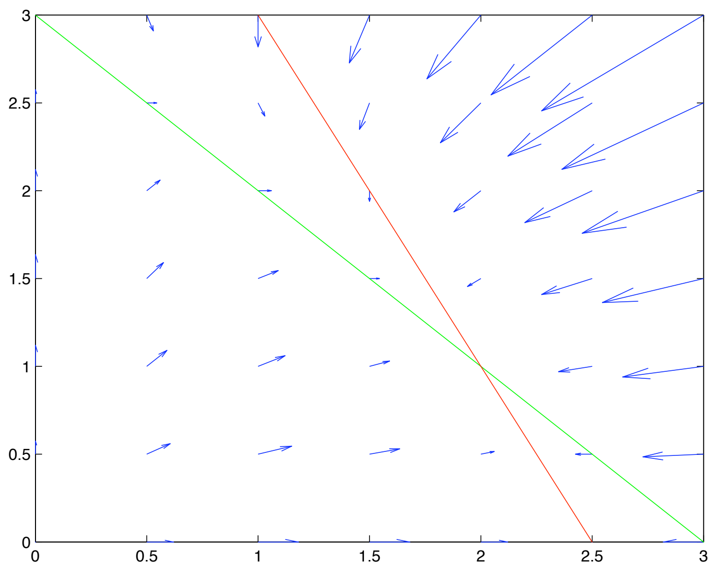


Figure 1: This is the phase plane for the competition system in question 6.

Question 7: For each of the following three systems of differential equations, determine whether all solutions converge to zero or not.
System 1:

$$\begin{aligned} \frac{d}{dt}x(t) &= x(t) + 4y(t), \\ \frac{d}{dt}y(t) &= 3x(t) - 3y(t). \end{aligned}$$

System 2:

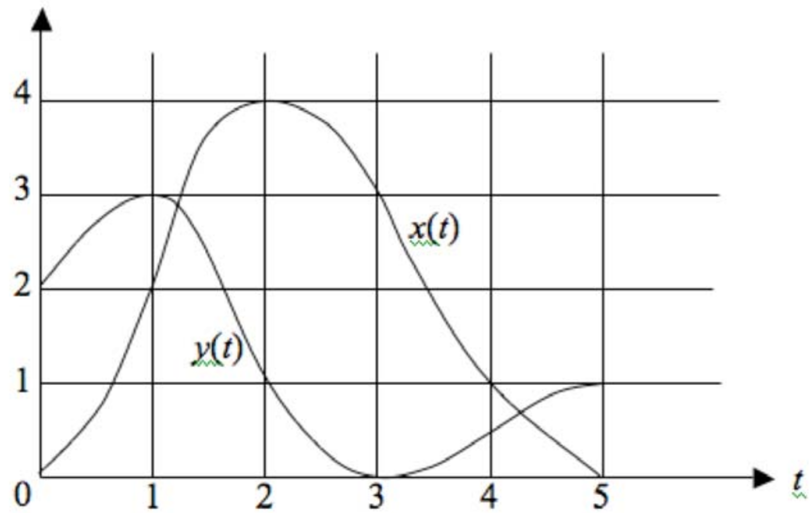
$$\begin{aligned} \frac{d}{dt}x(t) &= 2x(t) - 5y(t), \\ \frac{d}{dt}y(t) &= x(t). \end{aligned}$$

System 3:

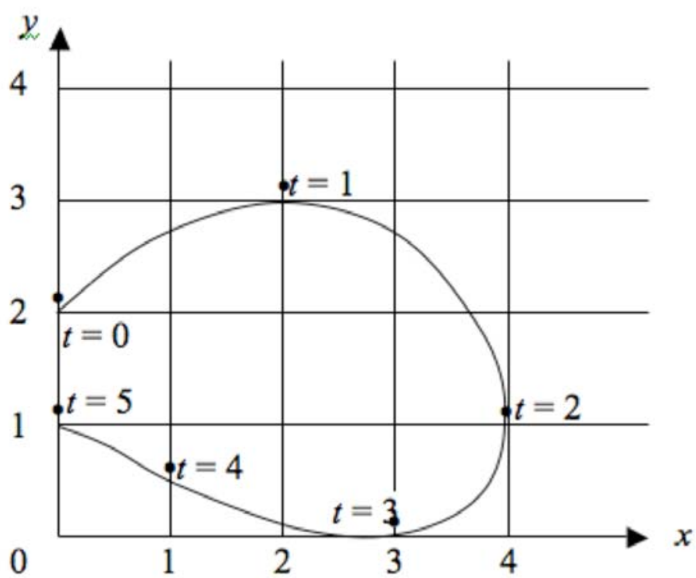
$$\begin{aligned}\frac{d}{dt}x(t) &= 2x(t) + 5y(t),\\ \frac{d}{dt}y(t) &= x(t) - 8y(t).\end{aligned}$$

Solution: For system 1, the matrix has the eigenvalues $\lambda = 3$ and $\mu = -5$. Since one of the two is positive, not all solutions converge to zero.
For system 2, the matrix has the eigenvalues $\lambda = 1+\mathrm{i}$ and $\mu = 1-\mathrm{i}$. Since the real part of the two eigenvalues is positive, no solution converges to zero.
For system 3, the matrix has the eigenvalues $\lambda = -1$ and $\mu = -5$. Since both eigenvalues are negative, all solutions converge to zero.

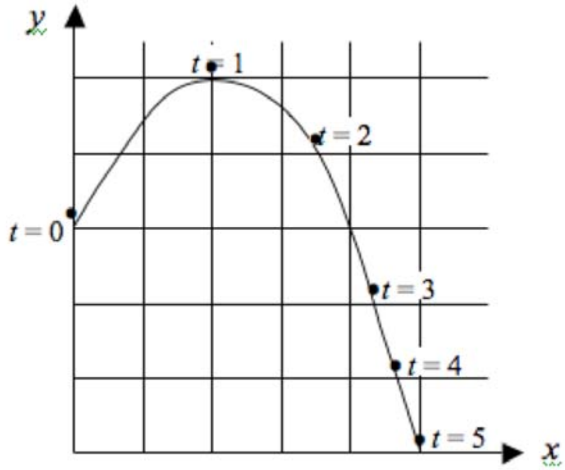
Question 8:
(a) Suppose the following are graphs of the solution of a system of two autonomous differential equations. Draw the phase-plane trajectory of this solution.



Solution:



(b) Suppose the following is the phase-plane trajectory of the solution of a system of two autonomous differential equations. Draw the graphs of this solution.



Solution: